Time-optimal Persistent Homology Representatives

for Data Growing Over Time

António Leitão

Joint work with Nina Otter

Scuola Normale Superiore, Pisa
Datashape, INRIA, Saclay

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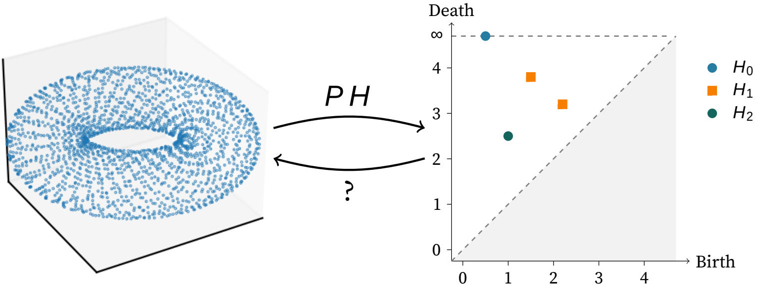
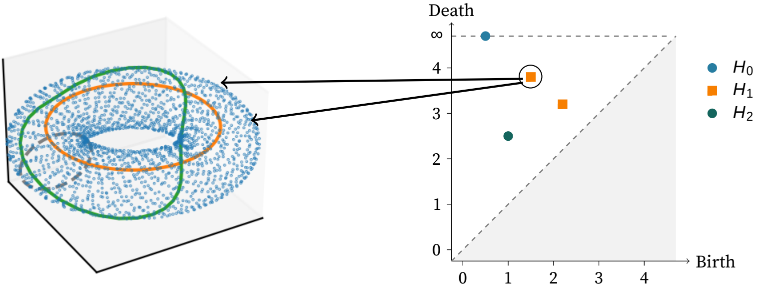
ComPer 2025: Workshop on Computational Topology

Motivation



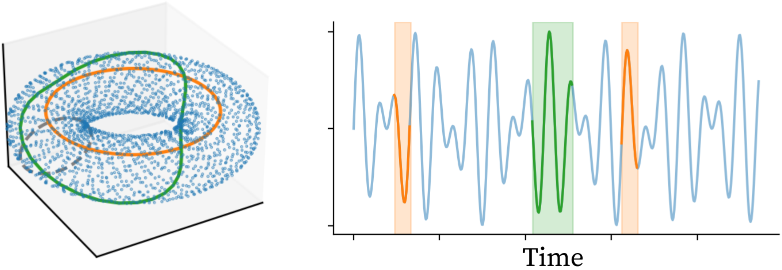
How can we better interpret
persistent homology information?

Motivation



Each homology class has
multiple representatives

Motivation



With time-indexed data we would prefer representatives that are *"close"* in time

Previous Work

Optimal Cycles

1. Localizing homology classes (Chen & Freedman 2008)
2. $_0$ minimization is NP-hard (Chen & Freedman 2010)
3. Linear Programming $_1$ norm (Dey et al. 2011)
4. Volume-optimal cycles (Obayashi 2018)

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Filtrations

1. Optimal filtered cycle bases (Escolar & Hiraoka 2016)
2. Filtered 1-cycles is NP-hard (Dey, Hou & Mandal 2019)
3. LP solutions almost always yield $\{-1, 0, 1\}$ coefficients (Li et al. 2021)

Previous Work

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Others

1. Harmonic (Basu & Cox 2021) Stable volumes (Obayashi 2021) Zigzag (Dey, Hou & Morozov 2024)

Outline

1. Representative Cycles
2. Linear Programming
3. Time Optimality
4. Examples
5. Applications
    - 5.1 Climate Data
    - 5.2 Transaction Networks

Representative Cycles

K: Simplicial Complex

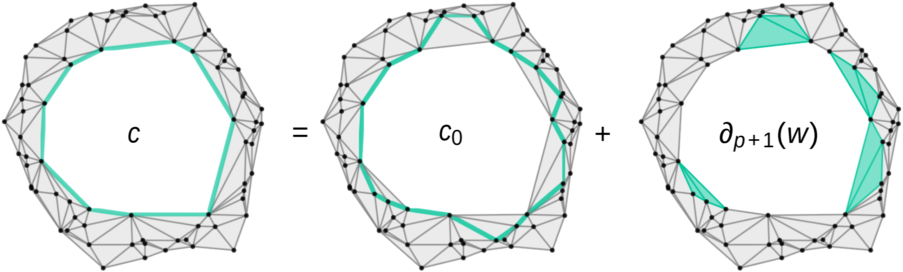
$S_p(K)$ its set of p-simplices

$C_p(K)$ vector space over F

$\partial_p : C_p(K) \longrightarrow C_{p-1}(K)$

p-cycles: elements of kernel of ∂_p

Representative Cycles



Representative Cycles

$$
\begin{aligned}
\text{min} \quad & (c) \\
\text{subject to} \quad & c = c_0 + \partial_{p+1}(w) \\
& w \quad C_{p+1}(K)
\end{aligned}
$$

Search for homologous p-cycles
by adding boundaries of $p + 1$-simplicies.

Linear Programming

$$
\begin{aligned}
\min \quad & \|W\mathbf{c}\|_1 = \sum_i \sum_j w_{ij}(c_j^+ + c_j^-) \\
\text{subject to} \quad & (\mathbf{c}^+ - \mathbf{c}^-) = \mathbf{c_0} + \partial_{p+1}(\mathbf{w}) \\
& \mathbf{c}^+, \mathbf{c}^- \geq 0
\end{aligned}
$$

Dey, Hirani, and Krishnamoorthy 2010
Li et al. (2021), *"Minimal Cycle Representatives in Persistent Homology Using Linear Programming: An Empirical Study With User's Guide"*

Linear Programming

$$\begin{pmatrix} w_{1,1} & 0 & 0 & \cdots & 0 \\ 0 & w_{2,2} & 0 & \cdots & 0 \\ 0 & 0 & w_{3,3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & w_{n,n} \end{pmatrix} \left(\begin{pmatrix} c_0^+ \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ c_k^- \\ 0 \end{pmatrix} \right)$$

Diagonal Matrix

w_{ii} = cost of having σ_i in the chain (e.g area/length)

$w_{ii} = 1 \approx$ minimizing the count

Linear Programming

$$\begin{pmatrix} w_{1,1} & w_{1,2} & w_{1,3} & \cdots & w_{1,n} \\ w_{2,1} & w_{2,2} & w_{2,3} & \cdots & w_{2,n} \\ w_{3,1} & w_{3,2} & w_{3,3} & \cdots & w_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ w_{n,1} & w_{n,2} & w_{n,3} & \cdots & w_{n,n} \end{pmatrix} \left(\begin{pmatrix} c_0^+ \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ c_k^- \\ 0 \end{pmatrix} \right)$$

Full weight matrix

w_{ij} = cost of having both σ_i and σ_j together in the chain.

Filtered Cycle Representative

What if we have a filtered simplicial complex?

Filtered Cycle Representative

$$\begin{aligned} \min \quad & (c) \\ \text{subject to} \quad & c = c_0 + \partial_{p+1}(w) \\ & w \quad C_{p+1}(K) \end{aligned}$$

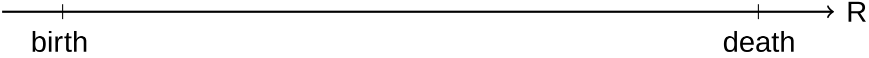
Li et al. (2021), *"Minimal Cycle Representatives in Persistent Homology Using Linear Programming: An Empirical Study With User's Guide"*
Chen and Freedman (2008), *Quantifying Homology Classes*

Filtered Cycle Representative

$$
\begin{aligned}
\min \quad & (c) \\
\text{subject to} \quad & c = c_0 + \partial_{p+1}(w) \\
& w \quad C_{p+1}(K) \\
& \text{Birth}(c) = a \\
& \text{Death}(c) = b
\end{aligned}
$$

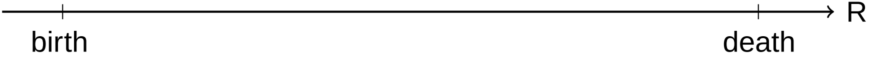
Li et al. (2021), *"Minimal Cycle Representatives in Persistent Homology Using Linear Programming: An Empirical Study With User's Guide"*
Chen and Freedman (2008), *Quantifying Homology Classes*

K_r $K_{r'}$ $\cdots$



We do not consider simplicies
born after the birth of the class

K_r K_{r^0} $\cdots$



$$P = f \quad 2 \; S_p(K_b) \; j \; \text{birth}(\quad) \quad bg$$

$$Q = f \quad 2 \; S_{p+1}(K_b) \; j \; \text{birth}(\quad) \quad bg$$

Filtered Cycle Representative

$$
\begin{aligned}
\min \quad & \|Wc\|_1 = \sum_i \sum_j w_{ij} (c_j^+ + c_j^-) \\
\text{subject to} \quad & (c^+ - c^-) = c_0 + \partial_{p+1}[P;Q](w) \\
& w \in \mathbb{R}^{|Q|} \\
& c \in \mathbb{R}^{|P|} \\
& c^+, c^- \geq 0
\end{aligned}
$$

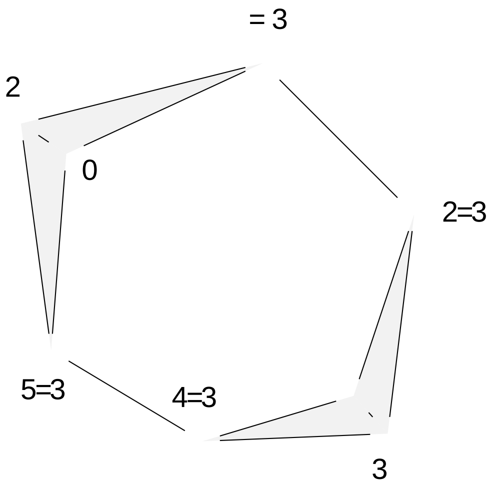
Dey, Hirani, and Krishnamoorthy (2010), Optimal homologous cycles, total unimodularity, and linear programming

Li et al. (2021), Minimal Cycle Representatives in Persistent Homology Using Linear Programming: An Empirical Study With User's Guide

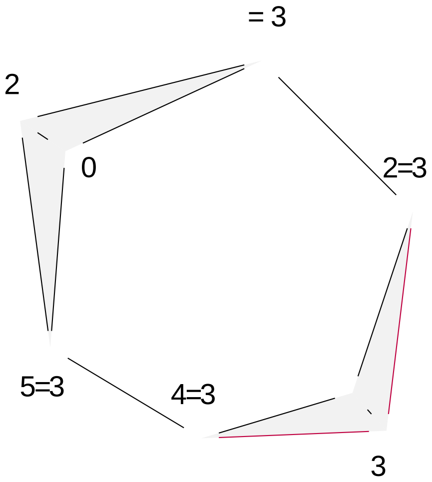
Time Optimality

How to measure how spread out representatives are in time

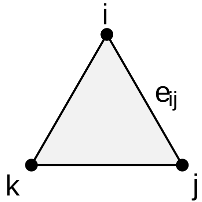
What if we have time data



What if we have time data



Time at Vertex-Level

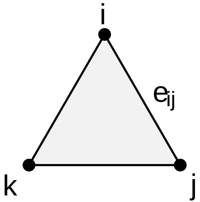


Cost of selecting e_{ij}

$$w_{e_{ij}} = |T_i - T_j|$$

We find chains whose vertices have
time labels that are close to each other

Time at Vertex-Level

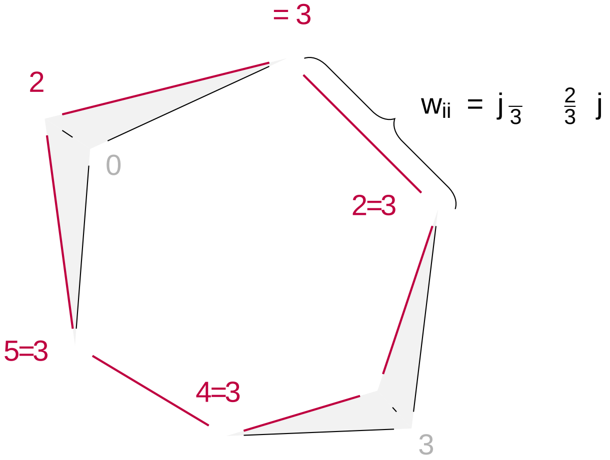


Cost of selecting

$$w = \max_{r \in \{i,j,k\}} T_r - \min_{s \in \{i,j,k\}} T_s$$

We find chains whose vertices have time labels that are close to each other

Time at Vertex-Level

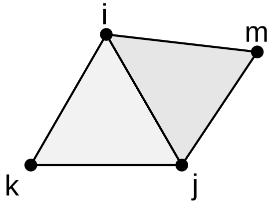


Time at Vertex-Level

$$W_{vertex}$$

$$\begin{bmatrix} w_{1;1} & 0 & 0 & & 0 \\ 0 & w_{2;2} & 0 & & 0 \\ 0 & 0 & w_{3;3} & & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & & w_{n;n} \end{bmatrix}$$

Time at Simplex-Level

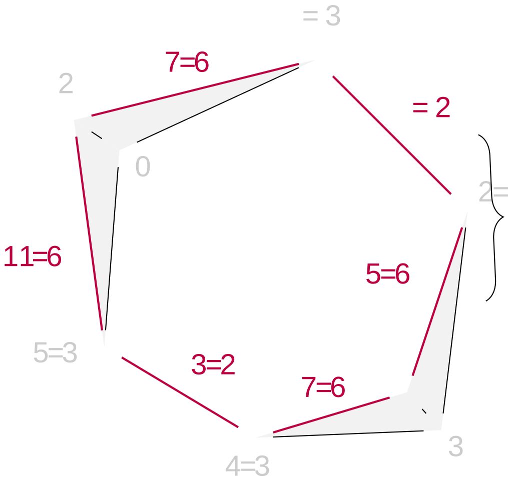


Cost of selecting both and

$$w_{;} = jT \quad T\; j$$

We rst situate each simplex in time
and nd a chain of closely situated simplices

Time at Simplex Level



$$w_{ij} = j\,\frac{}{2}\quad\frac{5}{6}\;j$$

Time at Simplex Level

$$W_{simplex}$$

$$\begin{bmatrix} w_{1;1} & w_{1;2} & w_{1;3} & & w_{1;n} \\ w_{2;1} & w_{2;2} & w_{2;3} & & w_{2;n} \\ w_{3;1} & w_{3;2} & w_{3;3} & & w_{3;n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ w_{n;1} & w_{n;2} & w_{n;3} & & w_{n;n} \end{bmatrix}$$

Example

Noisy sine with 4 revolutions

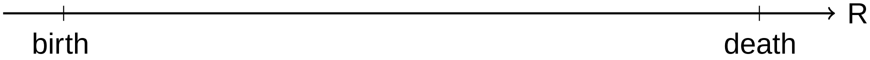
Simplex

Vertex

Edge Length

So what happened?

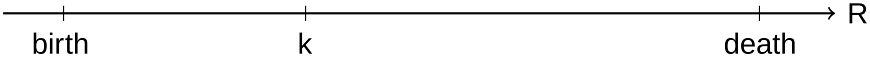
We have to increase our search space



Relaxing Persistence Constraint

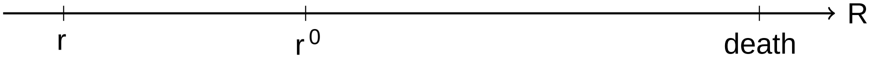
Increases search space

Makes sense when optimization is not area



Resulting persistence is death   k

K_r $K_{r'}$ $\cdots$



$$r \le r' \Rightarrow \min\{f(c) \mid c \in C_p(K_r)\} \le \min\{f(c) \mid c \in C_p(K_{r'})\}$$

Simplex

Vertex

Edge Length

Representative has 95% of total persistence

Simplex

Vertex

Edge Length

Method	L_1	Time Dispersion
Initial Representative	9:03	370
Time (Simplex)	5:51	112
Time (Vertex)	7:78	81
L_1 Optimal	4:93	178

Example

Network with time weighted edges

Example (Time Network)

Simplex

L 0 (Count)

Example (Time Network)

Simplex

L_0 (Count)

Method	Cycle Length	Time Dispersion
Initial Representative	75	255:69
Time-Optimal	26	12:24
L_0 Minimisation	24	20:15

Example

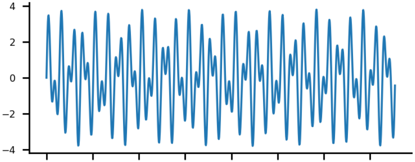
Quasi periodic signal

$$f(t) = 2\sin t + 1.8\sin\sqrt{3}t \quad 0 \le t \le 60\pi$$

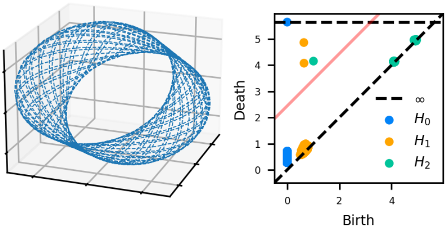
Gakhar and Perea (2023), *“Sliding Window Persistence of Quasiperiodic Functions”*

Example (Quasi-Periodic Signal)

$$f(t) = 2\sin t + 1.8\sin\sqrt{3}t \quad 0 \le t \le 60\pi$$



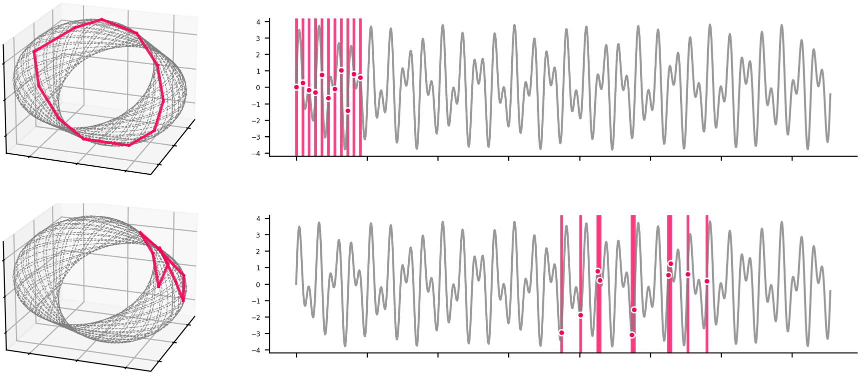
Time-series



Sliding window Embedding

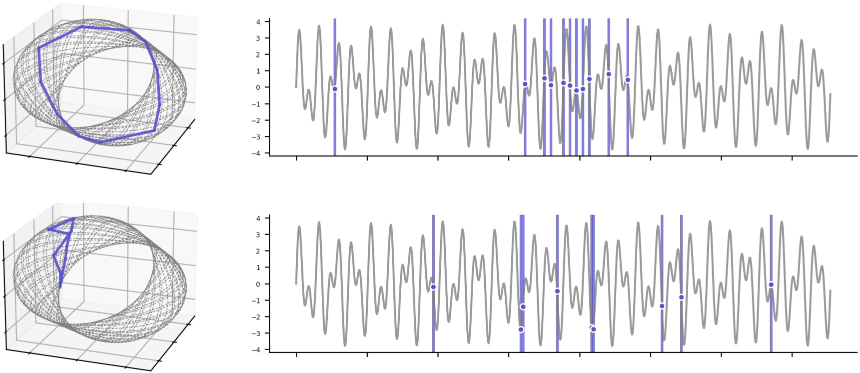
Gakhar and Perea (2023), *"Sliding Window Persistence of Quasiperiodic Functions"*

Example (Quasi-Periodic Signal)



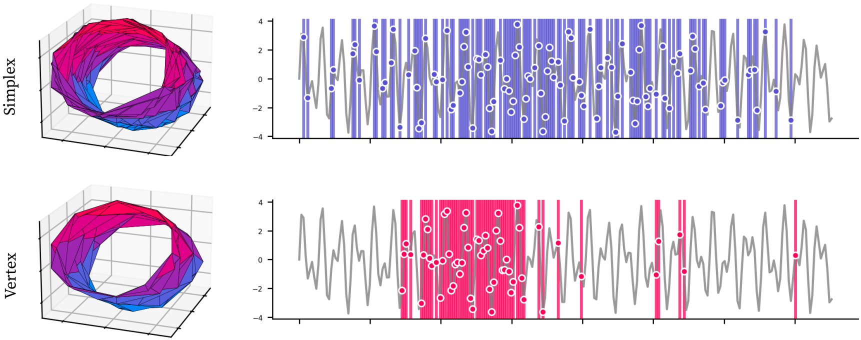
Vertex Based

Example (Quasi-Periodic Signal)



Simplex Based

Example (Quasi-Periodic Signal)

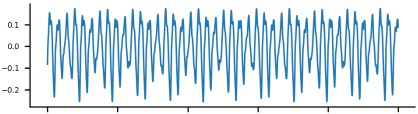


Application

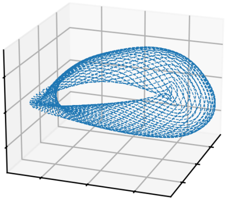
El Niño Southern Oscillation
climate model (ENSO)

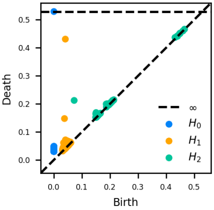
Christensen et al. (2017), *"Stochastic Parameterization and El Niño–Southern Oscillation"*

Application (El Niño)



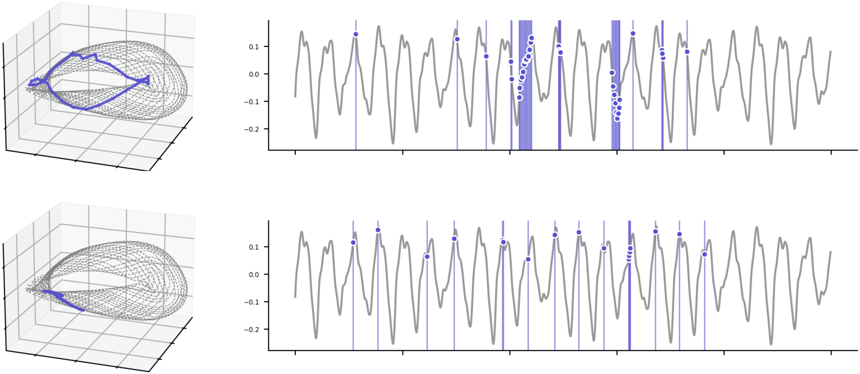
Time-series





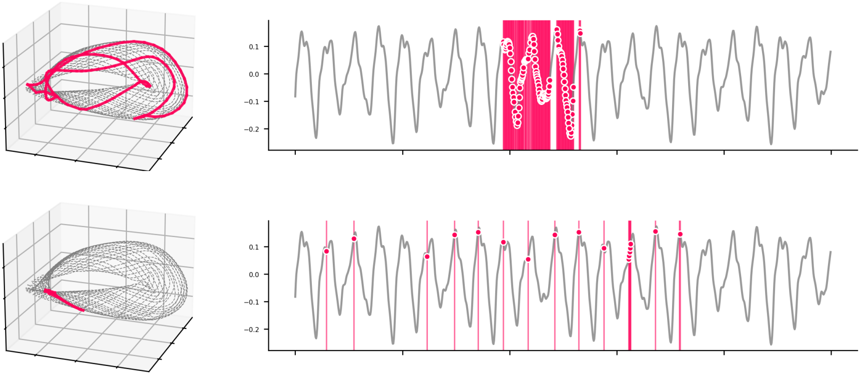
Sliding window Embedding

Application (El Niño)



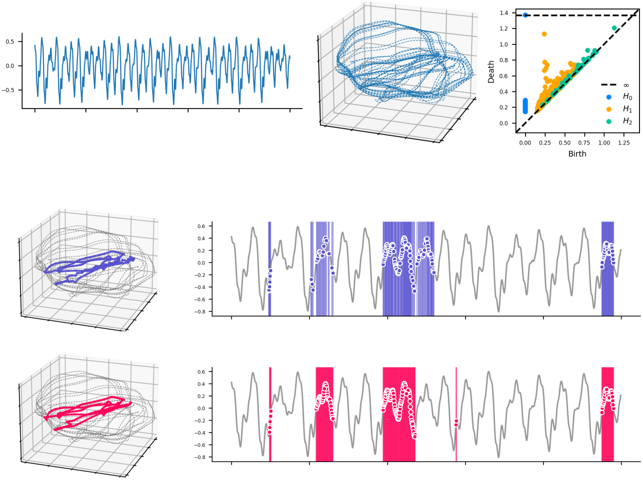
Simplex Based

Application (El Niño)



Vertex Based

Application (El Niño)



Application

Bank transaction networks
for credit fraud detection

Altman et al. (2024), *Realistic Synthetic Financial Transactions for Anti-Money Laundering Models*

Example (Time Network)

Example (Time Network)

Future Directions

Thank You

In collaboration with Nina Otter
Special Thanks to Hannah Christenssen