Colloquium

RAUL CURTO
University of Iowa

MOMENT-THEORETIC TECHNIQUES IN THE ANALYSIS OF FINITE ALGEBRAIC VARIETIES

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ABSTRACT. For a degree $2n$ real d-dimensional multisequence $\beta \equiv \beta^{(2n)} = \{\beta_i\}_{i \in Z_+^d, |i| \leq 2n}$ to have a representing measure μ, it is necessary for the *associated moment matrix* $M(n)$ to be positive semidefinite, and for the *algebraic variety* associated to β, V_β, to satisfy rank $M(n) \leq$ card V_β as well as the following *consistency condition*: if a polynomial $p(x) \equiv \sum_{|i| \leq 2n} a_i x^i$ vanishes on V_β, then $p(\beta) := \sum_{|i| \leq 2n} a_i \beta_i = 0$. In previous joint work with L. Fialkow and M. Möller, we proved that for the *extremal* case (rank $M(n) =$ card V_β), positivity and consistency are sufficient for the existence of a (unique, rank $M(n)$-atomic) representing measure.

In subsequent joint work with Seonguk Yoo, we considered cubic column relations in $M(3)$ of the form (in complex notation) $Z^3 = itZ + u\bar{Z}$, where u and t are real numbers. For (u,t) in the interior of a real cone, we proved that the algebraic variety V_β consists of exactly 7 points, and we then applied the above mentioned solution of the extremal moment problem to obtain a necessary and sufficient condition for the existence of a representing measure. This required a new representation theorem for sextic polynomials in Z and $\bar{Z}$ which vanish in the 7-point set V_β. Our proof of this representation theorem relies on two successive applications of the Fundamental Theorem of Linear Algebra.

More recently, for $n \in \mathbb{N}$, with S. Yoo we have considered the algebraic variety $\mathcal{V}$ obtained by intersecting $n+1$ algebraic curves of degree n in $\mathbb{R}^2$, when the leading terms of the associated bivariate polynomials are all different. We have obtained a new proof, based on the Flat Extension Theorem from the theory of truncated moment problems, that the cardinality of $\mathcal{V}$ cannot exceed $\binom{n+1}{2}$. In some instances, this provides a slightly better estimate than the one given by Bézout's Theorem. Thus, we have found a moment-theoretic approach to estimate the cardinality of certain algebraic varieties. Our main results on this topic contribute to the growing literature on the interplay between linear algebra, operator theory, and real algebraic geometry.